TERTIARY STUDENTS' UNDERSTANDING OF SECOND ORDER RELATIONSHIPS IN FUNCTION NOTATION

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Abstract
The function concept permeates many aspects of the mathematics curriculum at both the secondary and tertiary levels. Students' use and conceptualization of the function concept have been the subject of a number of research studies (see for example, Arnold (1992), Barnes (1988)). These studies have found that the overwhelming view students hold of this concept is that a function is a rule of correspondence, or more precisely, an algebraic formula, into which values are substituted. However, this rather narrow interpretation begins to breakdown when questions that require the use of more advanced reasoning skills are encountered. The competent use of function notation, given this stance, relies heavily on students' understanding of the symbolism involved. Thus the notion of a variable and how different variables relate are critical factors in students' ability to use the function concept purely within an algebraic context. The purpose of this study was to explore this feature by examining tertiary students' responses to a series of questions, in which relationships between variables were set within the framework of function notation. To assess such responses, the SOLO Taxonomy was used. The study has highlighted the difficulties students experience with second order relationships in non-routine function questions and the value of the SOLO Taxonomy in interpreting students' responses.

Introduction

Several research studies have been directed at the understanding of the function concept held by senior secondary and tertiary level students. Arnold (1992) and Barnes (1988), in particular have highlighted a number of problem areas Australian students have with this concept. Findings from these and other related studies conducted internationally (see for example Dreyfuss and Eisenberg (1982)), have shown that the majority of students believe a function needs to be represented by a formula, with few students identifying this concept with its geometrical applications.

Underpinning this algebraic interpretation of functions is the understanding of the symbolism associated with it. The formal $f(x)$ notation, while succinctly describing a wealth of information, causes many difficulties for students, as the implications of this notation are not fully realised. Deficiencies in this area can in part be explained by students use and interpretation of the literal symbols involved. Attempts at further clarification of these effects have been made by the authors (Pegg and Coady 1993, and Coady and Pegg in press). They have identified characteristics of higher-order algebraic skills needed for such concepts by classifying student responses to questions involving the notion of a variable using the SOLO Taxonomy (see Biggs and Collis (1982, 1991) and Pegg (1992) for further information). As a result of these investigations, the authors were able to take the initial definitions of the two relevant modes, namely Concrete Symbolic and Formal and identify the characteristics of these modes in relation to algebraic thinking. These findings can be summarised as follows:

Concrete Symbolic responses rely heavily on concrete referents, whether they be by experience or observation. In the case algebra, students responding in this mode, depend exclusively on the manipulation of symbols, using techniques such as factorisation, simplification, numerical substitution and the like. This generally means students make relatively quick closures, which are not contested even if the requirements of the question are not satisfied. A lack of consideration of alternatives is also a feature of the responses within this mode.

Formal responses were originally defined in terms of the ability to extract abstract concepts from generalisations. In the case of algebra, this allows students the freedom to hypothesise, challenge and draw conclusions, while simultaneously considering the relationship between the variables inherent in the question. Overriding generalisations between the variables are identified and used explicitly.

The purpose of this current study is to extend these findings by focussing on relationships between variables as they relate to function notation. Empirical evidence is also sought which clearly distinguishes the differences between Concrete Symbolic and Formal responses and provides, if possible, examples of a unistructural-multistructural-relational learning cycle in the Formal mode.

Methodology

To address the research issues mentioned above, a series of questions involving the concept of a variable within the context of function notation was given to 147 first-year university students. The ages of these students ranged from 17 - 20 years, with the majority of students ($\approx$ 80%) having studied calculus in their mathematics course in their final two years of secondary school. All students tested were enrolled in mathematically based undergraduate programs. Two questions are discussed in detail below, as they are representative of the questions asked. The first question consisted of two parts. Part i) was designed as a jargon-free version of a 'routine' function notation question. Part ii) was the equivalent of part i) but was couched in standard terminology. Question 2, while maintaining the general rubric of the function concept, requires an understanding which exceeds that needed in routine procedures.

Question 1: i) If y is increased by t, find an expression for $3y^2 + 2y$

Categorisation of responses reflected students' ability to translate the English phrase 'y is increased by t' into symbols. Two distinct groups of responses were identified which were subsequently coded as Concrete Symbolic and Formal. Responses classified as Concrete Symbolic could not successfully make this translation and instead focussed on manipulative techniques. Formal responses, on the other hand, indicated that students could make this translation. Students' responses were equally divided between the two modes.

Concrete Symbolic Responses

Responses within this mode fell into two groups, with both groups ignoring the phrase 'y is increased by t'. Incorrect manipulative procedures were used to formulate an answer. The first group consisted of two types of responses in which the expression $3y^2 + 2y$ was treated as either a single term or as two independent terms, $3y^2$ and $2y$. In each case, t was simply added or multiplied as the students saw fit. For example:

Type 1	Type 2
1. "$3y^2 + 2y + t$"	1. "$3(y^2 + t) + 2(y + t)$"
2. "$t(3y^2 + 2y)$"	2. "$3(yt)^2 + 2yt$

The second group of responses lapsed into pseudo equation-solving techniques, such as:

1. "$y + t = 3y^2 + 2y$
   $= 3y^2 + 2y + t$
   $= 0$"

2. "$t = 3y^2 + 2y$"

Formal responses

Responses in this mode hinged upon students paraphrasing 'y is increased by t' into symbols and substituting for y in the expression $3y^2 + 2y$. All students responding in this manner did so correctly. Formal reasoning was thus indicated, but the written scripts did not provide sufficient detail to allow a further breakdown of responses.

Question 1 ii): Given $f(x) = -2x^2 + 3x$, find $f(x + h) =$

Analysis of this question in terms of the SOLO Taxonomy was difficult. Trying to separate rote learnt responses from those that showed students did understand the fundamental concepts of function notation proved impossible from the written scripts. (However, followup interviews conducted as part of a second stage of this investigation (not reported here) showed that students found it easier to substitute the expression $x + h$ for x, with explanations similar to "Wherever you see and x in here [referring to $f(x) = -2x^2 + 3x$], you put in x + h.")

More students (n = 90)were able to answer this part as opposed to part i)(n = 77). This suggests interpretation difficulties rather than manipulative skills were probably of some concern for students. It could be argued that the reasoning skills needed to translate English into symbols, as in part i), are of a higher order than those required in routine substitutions, as in part ii). Even with this consideration in mind, the information provided by the written scripts did not allow for a legitimate appraisal to be made with the SOLO Taxonomy.

Question 2: If $f(1) = 5$ and $f(x + 1) = 2f(x)$, find the value of $f(3)$

The function concepts in this question required the use of higher level analytical skills to successfully complete this question. Two distinctive groups were identified which were then coded as representative of the Concrete Symbolic and Formal modes. The percentage of responses in each mode was 95% and 5% in the Concrete Symbolic and Formal modes, respectively.

Concrete Symbolic Responses

Responses at this level indicated that students had no real depth to their knowledge and understanding of function notation. If more than superficial substitution of numbers or letters was required, these students were completely lost. For example, one group consisted of responses in which students immediately substituted $x = 3$ and then, in some cases made some extraordinary calculations, reflecting little understanding or recognition of the interrelationships within the system. For example:

1. "$f(3 + 1) = 2f(3)$
   $\frac{f(4)}{2} = f(3)$"

2. "$f(x) = \frac{f(x+1)}{2}$
   $f(1) = \frac{f(1+1)}{2} = f(1) = 5$
   $f(3) = \frac{f(3+1)}{2} = f(2)$"

Other responses at this level concentrated purely on manipulative strategies that were incorrect. It was obvious from the way function notation was used that students lacked any real understanding and the symbols used were meaningless in their traditional sense. For example:

1. "$f(1) = 5$

$$f(x + 1) = 2f(x)$$
$$f(x + 1) = 10$$
$$x + 1 = 10$$
$$x = 9$$
$$\therefore f(3) = ?"$$

2. "$f(x) = \dfrac{f(x+1)}{2}$ $\quad 2x = x + 1$

$$1 + 1 = 2 \qquad x = 1$$
$$f(2) = 6 = x + 1 = 5"$$

These two examples typify responses reliant on transformation procedures and clearly demonstrate students' lack of expertise with function notation. In both instances, students chose to work outside the function context, preferring instead to substitute and then work with a more familiar system, that of solving equations. Once a result from the computations was achieved, function notation was recalled, with the answer being an unsuccessful attempt at combining both concepts.

In summary, Concrete symbolic responses were characterised by 'quick' substitutions. In general, they showed that students had removed their computations from the function notation context altogether, preferring to work within what was a more recognisable system to them.

Formal Responses

Responses classified as Formal indicated that students had a deeper knowledge of the concepts involved in function notation that went beyond the simple rearrangement of the formula or the substitution of numbers. However, it was only at the relational level that students were able to coordinate all aspects and respond fully to the question.

Unistructural responses: A response at this level showed that students could work within the function notation framework, but with one aspect only. For example:

$$"f(1) = 5$$
$$f(x + 1) = 2f(x)$$
$$f(2) = 2 \times 5 = 10"$$

Answers in this category were typically brief and to the point with students unable to continue after this computation had occurred.

Multistructural responses: Responses at this level showed that students could work within the structure afforded by function notation. However, they were characterised by a number of seemingly independent operations with no overall plan evident. Students appeared to choose steps almost at random and while in many cases, these were valid, they achieved no logical purpose for answering the question. For example:

1. "$f(x + 1) = 2f(x)$
$$\begin{aligned} &\text{let } x = 0 \\ &f(1) = 2f(0) \\ &2f(0) = 5 \\ &f(0) = 2\frac{1}{2} \\ &f(3) = 7\frac{1}{2}\text{"} \end{aligned}$$

2. "$f(1) = 5$
$$\begin{aligned} f(x + 1) &= 2f(x) \\ f(2 + 1) &= 2f(2) \\ f(3) &= 2f(2)\text{"} \end{aligned}$$

Relational responses: Relational responses indicated that the students were capable of using the concepts underlying function notation and could note and use the interrelationships existing within the question. Using these concepts and relationships allowed the correct answer to be reached:

$$\begin{aligned} \text{"}f(1+1) &= 2f(1) \\ f(2) &= 10 \\ f(2 + 1) &= 2f(2) \\ f(3) &= 20\text{"} \end{aligned}$$

Students capable of responding at this level also had this procedure at their disposal:

$$\begin{aligned} \text{"}f(3) &= 2f(2) \\ &= 2[2f(1)] \\ &= 2[2 \times 5] \\ &=20\text{"} \end{aligned}$$

Implications and Conclusion

The general theme of this paper was the examination and classification of students' responses with the SOLO Taxonomy, to questions involving second order relationships among variables framed within function notation. Successful completion of each question required a response in the Formal mode, with Question 2 in particular, needing a relational level answer. The percentage of students correct in each question differed markedly, from 61% in Question 1 part ii), to an alarming low 5%

in Question 2. This result reflects poorly on students' ability to recognise and use second-order relationships inherent in the notion of a variable, when presented in function notation format.

Despite the different syntactic construction of each question, similarities were noted in the types of responses classified as Concrete Symbolic and Formal. Concrete symbolic levels of reasoning are restricted to the substitution of letters or numbers. This was particularly evident in Question 2 when meaningless substitution of numbers took place, with students tending to opt out of working within the function notation structure. Responses classified as Formal however, were able to work within this framework. Students tried to maintain the integrity of the function notation fabric, with some being more successful than others.

The findings of this study have also mirrored those found previously by the authors, which clearly show qualitatively different responses representative of Concrete Symbolic and Formal thinking, notwithstanding the shift in focus to function notation. Concrete Symbolic responses were characterised by a dependence on manipulative procedures. Processing of the question was confined to such procedures, with the resultant answers regularly bearing no relationship to the question. Resolution of this conflict was not addressed, with some students seemingly unaware of its existence in the first place.

In contrast to these responses, those classified as Formal verified that students did have these manipulative techniques at their disposal, but these were used as a 'tool' in determining the answer. That is, the relationships between the variables were the focal point and the appropriate manipulations needed were secondary in the solution process.

The SOLO Taxonomy has again proved beneficial in gauging the functional performance of a student. The majority of students in this sample were found to be functioning in the Concrete Symbolic mode. This mode is characterised by the almost exclusive use of transformation techniques, when responding to a question. Such reliance raises several important issues that warrant further investigation. These include, what level of competence is needed in the Concrete Symbolic mode before progression is made to the Formal mode, and associated with this, is the question of whether students can continue to develop very advanced skills in this mode without ever advancing to the Formal mode. Clearly answers to these concerns would greatly enhance our understanding of students' learning of algebraic concepts and add another dimension to the current debate surrounding the place of manipulative skills in the learning and teaching of algebra.

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